

A sufficient conditon for solvability of finite groups

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Abstract

The following theorem is proved: Let G be a finite group and $\pi_e(G)$ be the set of element orders in G . If $\pi_e(G) \cap \{2\} = \emptyset$; or $\pi_e(G) \cap \{3, 4\} = \emptyset$; or $\pi_e(G) \cap \{3, 5\} = \emptyset$, then G is solvable. Moreover, using the intersection with $\pi_e(G)$ being empty set to judge G is solvable or not, only the above three cases.

1 Introduction

Let G be a finite group. We have two basic sets: $|G|$ and $\pi_e(G)$. There are many famous works about $|G|$ in the history of group theory. The set $\pi_e(G)$ was studied first by author in [1]. The main results in [1] are:

Lemma 1.1 *Let G be a finite group. If $\pi_e(G) = \{1, 2, 3, 5\}$, then $G \cong A_5$.*

One can easily get the following conclusion from [2]:

Lemma 1.2 *Let G be a finite group with $\pi_e(G) = \{2, \text{the factors of } (2^n - 1) \text{ and } (2^n + 1), n \geq 2\}$. Then $G \cong L_2(2^n)$.*

For the simple group $Sz(2^{2m+1})$, we have the following result([3]):

Lemma 1.3 *Let G be a finite group. If $\pi_e(G) = \{2, 4, \text{the factors of } (2^{2m+1} - 1), (2^{2m+1} - 2^{m+1} + 1) \text{ and } (2^{2m+1} + 2^{m+1} + 1), m \geq 1\}$, then $G \cong Sz(2^{2m+1})$.*

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We will use the above three lemmas in the following discussion.

Compare with the study of $|G|$, we can also ask similar question about $\pi_e(G)$:

In [4], one can find the definition of h function: for a set S of element orders of a finite group, $h(S)$ is defined to be the number of non-isomorphic groups G with $\pi_e(G) = S$. There are many study about the h function with $h(S) = 1$, which means that the group G can be characterized by $\pi_e(G)$. The recent study can be find in [5].

Same as CLT-group ([6]), we study COE-group in [7].

As the solvability can be decided by the order of a group ([8]), we give condition about solvability by $\pi_e(G)$. This can be seen as a supplement of the following theorem in [9]:

Theorem 1.4 *Let G be a finite group and $\pi_e(G)$ be the set of element orders. Let $|\pi(G)|$ be the number of prime in $\pi_e(G)$ and $|\chi(G)|$ be the number of composite number in $\pi_e(G)$. Then $|\pi(G)| \leq |\chi(G)| + 3$, and G is simple if $|\pi(G)| = |\chi(G)| + 3$.*

Definition 1.5 *Let G be a finite group and $\pi_e(G)$ be the set of element orders of G . A set T is called IES if $T \cap \pi_e(G) = \emptyset$ implies G is solvable.*

If an IES-set T with $|T| = 1$, then by Feit-Thompson theorem, we get $T = \{2\}$. For the other set T , there are examples of non-solvable groups. So we get

For an IES-set, $|T| = 1$ if and only if $T = \{2\}$.

We can consider the IES-set T with $|T| = 2$. If $T = \{2, *\}$, where $*$ is an integer > 2 , we can get G is solvable and this is trivial. We need to consider the case that $T = \{3, **\}$, where $** > 3$.

J. G. Thompson classified all the minimal simple groups in [10]:

Lemma 1.6 *The minimal simple groups are:*

- (1) $L_2(p)$, $p > 3$, $5 \nmid (p^2 - 1)$, where p is a prime;
- (2) $L_2(2^p)$, where p is a prime;
- (3) $L_2(3^p)$, where p is a odd prime;
- (4) $L_3(3)$;
- (5) $Sz(2^{2m+1})$, where $2m + 1$ is a prime.

Notice that in the above lemma, the simple groups in (1)-(4) all have an element of order 3. For the simple group $Sz(2^{2m+1})$, since $\pi_e(G) = \{2, 4$, the factors of $(2^{2m+1} - 1)$, $(2^{2m+1} - 2^{m+1} + 1)$ and $(2^{2m+1} + 2^{m+1} + 1)$, $m \geq 1\}$, we get that $3 \notin \pi_e(Sz(2^{2m+1}))$,

but $4, 5 \in \pi_e(Sz(2^{2m+1}))(5 \mid (2^{2m+1} - 2^{m+1} + 1)(2^{2m+1} + 2^{m+1} + 1) \text{ for } 5 \mid (2^{4m+2} + 1))$. Hence if $|T| = 2$, then $T = \{3, 4\}$ and $\{3, 5\}$ are IES-sets. We claim that there are no other IES-set T with $|T| = 2$.

(a) Let $T = \{3, x\}$, where $x > 5$. Notice $\pi_e(Sz(2^3)) = \{1, 2, 4, 5, 7, 13\}$ and $\pi_e(Sz(2^5)) = \{1, 2, 4, 5, 25, 31, 41\}$. We get that the common element orders in the above two minimal simple groups are $\{1, 2, 4, 5\}$. So we can get a counterexample for any $x > 5$.

(b) Let $T = \{4, y\}$, where $y > 4$. Notice $\pi_e(A_5) = \{1, 2, 3, 5\}$. To exclude this case, we know $y = 5$. Since $\pi_e(L_2(2^3)) = \{1, 2, 3, 7, 9\}$, thus such y such that T is an IES-set does not exist.

(c) Let $T = \{5, z\}$, where $z > 5$. Since $\pi_e(L_2(2^3)) = \{1, 2, 3, 7, 9\}$ and $\pi_e(L_2(2^5)) = \{1, 2, 3, 11, 31, 33\}$, we get that such T does not exist by the same reason.

For the other T with $|T| = 2$, we can find a counterexample from $\pi_e(A_5) = \{1, 2, 3, 5\}$.

Next we consider the case of $|T| = 3$. Let $T = \{n_1, n_2, n_3\}$. From the above discussion, n_1, n_2, n_3 are odd. We can assume that $n_1 < n_2 < n_3$.

Suppose $n_1 = 3$. If n_2 or $n_3 \in \{4, 5\}$, it is a trivial consequence of the above discussion.

So we can assume that $n_2 > 5$ and $n_3 > 6$. Since $\pi_e(Sz(2^3)) = \{1, 2, 4, 5, 7, 13\}$, $\pi_e(Sz(2^5)) = \{1, 2, 4, 5, 25, 31, 41\}$ and $\pi_e(Sz(2^7)) = \{1, 2, 4, 5, 29, 113, 127, 145\}$, we can get an counterexample for any n_2, n_3 . Hence such T exists.

Suppose $n_1 = 4$. We can assume that $n_2 > 4$ and $n_3 > 5$. Since $\pi_e(L_2(2^3)) = \{1, 2, 3, 7, 9\}$, $\pi_e(L_2(2^5)) = \{1, 2, 3, 11, 31, 33\}$ and $\pi_e(L_2(2^7)) = \{1, 2, 3, 43, 127, 129\}$, we can get an counterexample for any n_2, n_3 . Hence such IES-set does not exist. We can also get the same conclusion for $n_1 = 5$.

For $n_1 > 5$, $\pi_e(A_5) = \{1, 2, 3, 5\}$ will give us an counterexample.

Therefore, the nontrivial IES-set T with $|T| = 3$ does not exist.

Finally, we consider the case of IES-set T with $|T| > 3$. We claim that no such IES-set exists. To do this, we give the following lemma.

Lemma 1.7 *Let m, n be two integer with $d = (m, n)$. Then $(2^m - 1, 2^n - 1) = 2^d - 1$.*

Proof: Suppose that $m = nq + r$, where $0 \leq r < n$. Then $2^m - 1 = 2^{nq+r} - 1 = 2^r(2^{nq} - 1) + 2^r - 1 \equiv 2^r - 1 \pmod{2^n - 1}$. So $(2^m - 1, 2^n - 1) = (2^r - 1, 2^n - 1)$. In this way, we can get

$$(2^m - 1, 2^n - 1) = 2^d - 1.$$

□

This lemma is a direct consequence of Theorem 1 of Section 7.4 in [11].

Corollary 1.8 *Let p, q be two different primes. Then $\pi_e(L_2(2^p)) \cap \pi_e(L_2(2^q)) = \{1, 2, 3\}$.*

Proof: It is well-known that $|L_2(2^p)| = 2^p(2^{2p} - 1)$, $|L_2(2^q)| = 2^q(2^{2q} - 1)$, and $(2p, 2q) = 2$. By Lemma 1.7, there is no common element order in $L_2(2^p)$ and $L_2(2^q)$ except 1, 2, 3. \square

Corollary 1.9 *Let p, q are two different odd primes. Then $\pi_e(Sz(2^p)) \cap \pi_e(Sz(2^q)) = \{1, 2, 4, 5\}$.*

Proof: We first prove that $(2^{2p} + 1, 2^{2q} + 1) = 5$ for any different odd prime p, q .

Clearly, $(2^{2p} + 1, 2^{2q} + 1) \mid (2^{4p} - 1, 2^{4q} - 1)$. By Lemma 1.7, $(2^{4p} - 1, 2^{4q} - 1) = 2^4 - 1 = 3 \times 5$. Since $3 \nmid 2^{2p} + 1$, $(2^{2p} + 1, 2^{2q} + 1) = 5$.

In the same way, we can get that $(2^{2p} + 1, 2^q - 1) = (2^{2q} + 1, 2^p - 1) = 1$.

It is easy to get that $\pi_e(Sz(2^p)) \cap \pi_e(Sz(2^q)) = \{1, 2, 4, 5\}$. \square

Now we continue the discussion of IES-set T with $|T| = s > 3$. Let $T = \{n_1, n_2, \dots, n_s\}$, where $n_1 < n_2 < \dots < n_s$ and $s > 3$.

As above, the case that $n_1 = 2$, or $\{n_1, n_2\} = \{3, 4\}$, or $\{n_1, n_2\} = \{3, 5\}$ is trivial. If $n_1 > 5$, A_5 will be a counterexample. So we need to consider the following cases:

(d) Let $T = \{3, n_2, \dots, n_s\}$, where $n_2 > 5$. Notice that $\pi_e(Sz(2^p)) \cap \pi_e(Sz(2^q)) = \{1, 2, 4, 5\}$. We get that there exists $k \in \pi_e(Sz(2^p)) - \{1, 2, 4, 5\}$ and $k \notin \pi_e(Sz(2^q))$. Hence for any $s = |T|$, there are enough large primes p to provide counterexample. Thus no such T exists.

(e) Let $T = \{4, n_2, \dots, n_s\}$, where $n_2 > 4$. Notice $\pi_e(L_2(2^p)) \cap \pi_e(L_2(2^q)) = \{1, 2, 3\}$. We get that there exists $k \in \pi_e(L_2(2^p)) - \{1, 2, 3\}$ and $k \notin \pi_e(L_2(2^q))$, $q \neq p$. Hence for any $s = |T|$, there are enough large primes p to provide counterexample. Thus no such T exists.

Hence we get:

Theorem 1.10 *Let G be a finite group and $\pi_e(G)$ be the set of element orders of G . If $2 \notin \pi_e(G)$, $\pi_e(G) \cap \{3, 4\} = \emptyset$, or $\pi_e(G) \cap \{3, 5\} = \emptyset$, then G is solvable. Furthermore, T is an IES-set if and only if $T = \{2\}$, $\{3, 4\}$ or $\{3, 5\}$.*

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References

- [1] Wujie Shi, A Characterization of A_5 (in Chinese), J. Southwest China Normal Univ. (Natural Soc.), 3(1986), 11-14.
- [2] Wujie Shi, The characterization of J_1 and $PSL_2(2^n)$ (in Chinese), Advance in Math. (China), 16:4(1987), 397-401.
- [3] Wujie Shi, A characterization of Suzuki's simple groups, Proc. Amer. Math. Soc., 114:3(1992), 589-591.
- [4] Wujie Shi, The finite groups with given set of element orders(in Chinese), Chinese Science Bulletin, 42:16(1997), 1703-1706.
- [5] A.V. Vasilev and M.A. Grechkoseeva, Recognition by spectrum for simple classical groups in characteristic 2, Siberian Math. J., 56:6(2015), 1009-1018.
- [6] J.F. Humphreys, On groups satisfying the convers of Lagranges theorem, Proc. Camb. Phil. Soc., 75(1974), 25-32.
- [7] Wujie Shi, Finite groups defined by the sets of their element orders, J. Southwest China Normal Univ. (Natural Soc.), 22:5(1997), 481-486.
- [8] Junhua He and Wei Pu, On the number n which makes any finite groups are solvable with order prime to n (in Chinese), J. Southwest China Normal Univ. (Natural Soc.), 24:6(1999), 612-614.
- [9] H. Deng and W. Shi, A simplicity criterion for finite groups, J. Algebra, 191:1(1997), 371-381.
- [10] J.G. Thompson, Nonsolvable finite groups all of whose local subgroups are solvable I, Bull. Amer. Math. Soc. 74(1968), 383-437.
- [11] Zhao Ke and Qi Sun, The Lecture of the Number Theory(in Chinese), The Second Edition, Part two, page 14.